Newton's Law of Gravitation

Newton concluded that the gravitational force between any two masses was proportional to the masses (make the masses bigger and the gravitational force between them will get bigger) and inversely proportional to the square of the distance between the center of mass of the two bodies. The proportionality constant is called the *universal gravitational constant "G,"* and the magnitude of the overall expression as an equality is:

$$F_g = G\frac{m_1 m_2}{r^2}$$

Historical Note: The way Newton determined that it was r squared, versus just r or r to the third, was very clever. His reasoning follows:

BACKGROUND:

The acceleration of a falling apple close to the earth's surface is $a_{earth} = 9.8 \text{ m/s}^2$, where the distance between the surface and the earth's center is approximately 4000 miles.

The acceleration of the moon around the earth's surface is $a_{earth} = 2.722\text{x}10^{-3} \text{ m/s}^2$, where the distance between the earth and moon is approximately 240,000 miles.

Note that he got this acceleration by noticing that the time it takes the moon to orbit the earth is approximately 28 days, and the acceleration is acceleration. That is, he observed that:

$$v_{moon} = \frac{\text{distance traveled in one orbit}}{\text{time for one orbit}} = \frac{2\pi R}{T} = \frac{2\pi(240{,}000 \text{ miles})(1609 \text{ meters/mile})}{(28 \text{ days})(24 \text{ hrs/day})(60 \text{ min/hour})(60 \text{ sec/min})} = 1003 \text{ m/s.}$$

Using the centripetal acceleration information, he additionally wrote:

$$a_{moon} = \frac{v^2}{R} = \frac{(1003 \text{ m/s})^2}{(240{,}000 \text{ miles})(1609 \text{ meters/mile})} = 2.6\text{x}10^{-3} \text{ m/s}^2.$$

Soooo, Newton's Second Law states the sum of the forces in a particular direction is proportional to the acceleration in that direction, so Newton could write:

$$F_{apple} \,\alpha\, a_{apple}$$
and
$$F_{moon} \,\alpha\, a_{moon}$$

Newton also surmised that the farther away the objects were, the small the gravitational force produced by each other (i.e., that the force was inversely proportional to some power of the distance r between the two). That is:

$$F_{apple} \,\alpha\, \frac{1}{r^n_{apple}}$$
and
$$F_{moon} \,\alpha\, \frac{1}{r^n_{moon}}$$

All of this allowed him to write:

$$a_{apple} \;\alpha\; \frac{1}{r^n_{apple}} \quad\Rightarrow\quad (9.8 \text{ m/s}^2) \;\alpha\; \frac{1}{(4000 \text{ mi})^n}$$
and
$$a_{moon} \;\alpha\; \frac{1}{r^n_{moon}} \quad\Rightarrow\quad (2.722\text{x}10^{-3} \text{ m/s}^2) \;\alpha\; \frac{1}{(240{,}000 \text{ mi})^n}$$

Dividing the top by the bottom of each side of the proportionality, we get:

$$\frac{(9.8 \text{ m/s}^2)}{(2.722\text{x}10^{-3} \text{ m/s}^2)} \;\alpha\; \frac{\left[\dfrac{1}{(4000 \text{ mi})^n}\right]}{\left[\dfrac{1}{(240{,}000 \text{ mi})^n}\right]}$$

$$\Rightarrow \quad 3600 \;\alpha\; 60^n$$

$$\Rightarrow \quad n = 2$$

This was the reasoning that led him to conclude that it was the inverse of the distance SQUARED that fit the force function (versus the inverse or the inverse cubed, etc.)

Interesting side note: When Newton first did this calculation, he got an "n" value that was not anywhere close to a whole number. Believing (evidently) that nature doesn't act that way, he put his theory aside as being incorrect. The problem was that the then accepted distance between the earth and moon was wildly incorrect. When a better measure of that number was published, he went back to his problem, but the more accurate value into his equation and came out with an "n" value that was very close to 2. That is when he embraced his theory as being most probably correct.

Kepler's Laws

Kepler's First: All planets move in elliptical orbits with the Sun at one of the focal points. (Law of orbits)

Kepler's Second: A line drawn from the Sun to any planet sweeps out equal areas in equal time intervals. (Law of areas.)

Kepler's Third: The square of the orbital period of any planet is proportional to the cube of the average distance from the planet to the Sun. (Law of period.)

Kepler's First: All planets move in elliptical orbits with the Sun at one of the focal points.

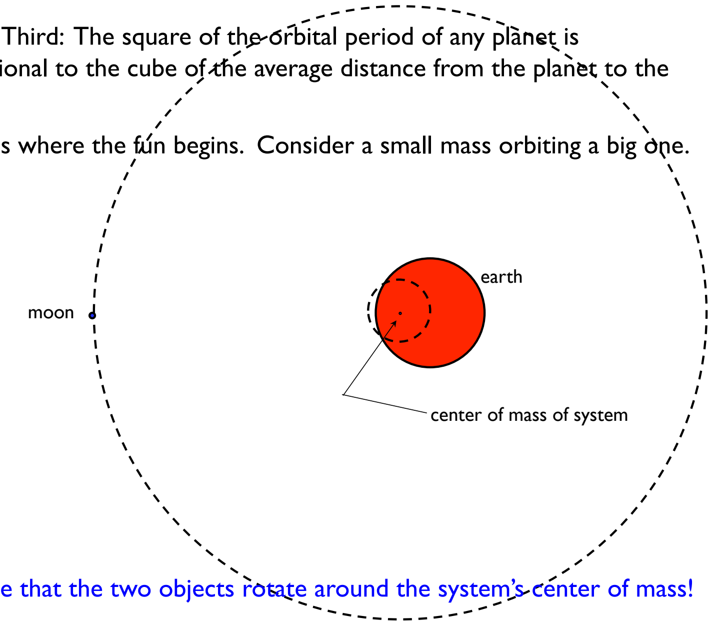
Using conservation of angular momentum and conservation of energy, it is possible to derive an expression for the radial position of a planet as a function of its angular position in the orbit (i.e., $r(\Theta)$). The derived expression that is that of an ellipse.

Kepler's Second: A line drawn from the Sun to any planet sweeps out equal areas in equal time intervals.

It turns out that the derivable expression for a planet's area sweep with time (i.e., dA/dt) looks just like the derived expression for a planet's angular momentum (give or take a constant). As the angular momentum of a torque-free body is constant, dA/dt must also be constant.

Kepler's Third: The square of the orbital period of any planet is proportional to the cube of the average distance from the planet to the Sun.

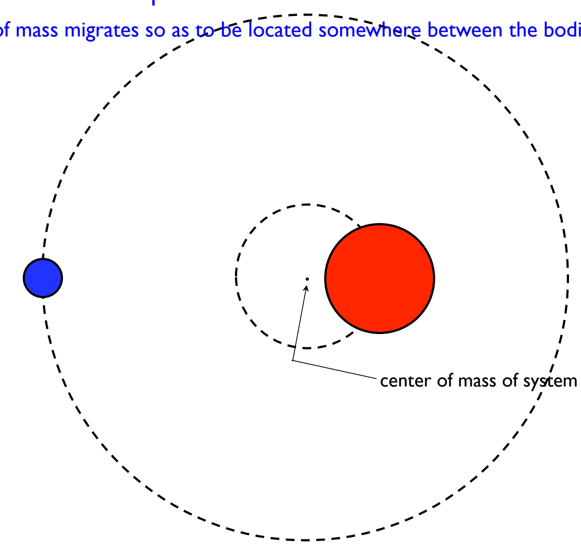
This is where the fun begins. Consider a small mass orbiting a big one.



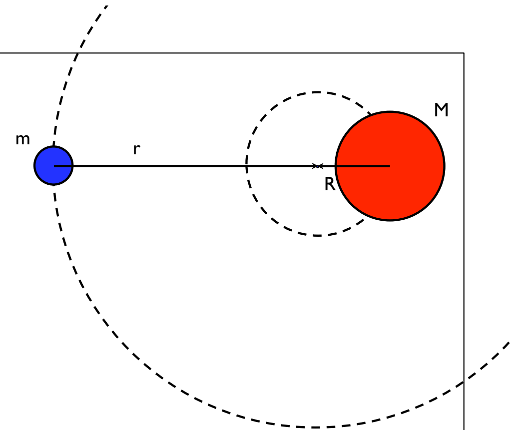
Notice that the two objects rotate around the system's center of mass!

What if the masses are comparable in size?

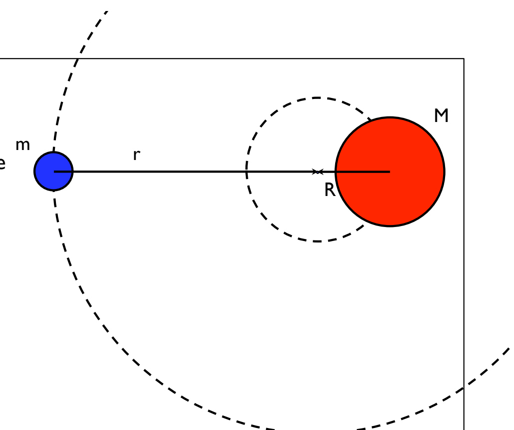
The center of mass migrates so as to be located somewhere between the bodies.



The set-up:



The math: We are looking at a gravitational force with a centripetal acceleration with the mass's velocity equal to the net distance traveled in one orbit (two pi r) divided by the orbital period (T). With all this, we can write:



$$\sum F_{centripetal}:$$

$$G\frac{mM}{(r+R)^2} = ma_c$$

$$= m\left(\frac{v^2}{r}\right)$$

$$= m\left(\frac{\left(\frac{2\pi r}{T}\right)^2}{r}\right)$$

$$\Rightarrow \quad T^2 = \left(\frac{4\pi^2}{GM}\right)\left(r(r+R)^2\right)$$

So what happens if "m" is very small in comparison to M (as in, satellite-like, or planet versus sun-like)? Then R goes to zero and the "period squared" expression becomes:

$$T^2 = \left(\frac{4\pi^2}{GM}\right)r^3$$

This is what Kepler observed in coming to his Third Law, and it turns out to be an approximation that is good only because our planets are small in comparison to the sun.

One more thing: We can rearrange the "period squared" expression to read:

$$GM = \left(\frac{2\pi}{T}\right)^2 r^3$$

Evidently, if we can determine the period (T) of two orbiting stars, then determine either the distance between the stars, we can determine the mass within the system. What's more, if one star is very massive while the other is not (a supergiant and a white dwarf?), we will know the mass of the more massive star.

Alternately, consider the mathematical manipulation shown to the right:

$$GM = \left(\frac{2\pi}{T}\right)^2 r^3$$

$$= \frac{(2\pi)^2}{T^2} r^3 \left(\frac{(2\pi)T}{(2\pi)T}\right)$$

$$= \left(\frac{(2\pi r)^3}{T^3}\right)\left(\frac{T}{(2\pi)}\right)$$

$$= v^3\left(\frac{T}{(2\pi)}\right)$$

What it suggests is that you can also determine the mass within a binary system if you can determine the period (T) of the two orbiting stars along with the *velocity* of their orbital motion.